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Numerical modelling of flux spectra formation for surface gravity waves

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By means of direct numerical solution of the kinetic equation for surface gravity waves, it is shown that under certain conditions the constant flux spectra of nonlinear waves, first predicted by Zakharov & Filonenko (1966) for an infinite frequency domain, can be formed in a finite frequency interval. For the case of angular isotropic spectra the conditions and timescales of this flux spectra formation are evaluated.

1. Introduction

As first shown by Hasselmann (1962) and confirmed later by Zakharov & Filonenko (1966), the temporal evolution of a wave-action wavenumber spectrum $n(\boldsymbol{k})$ for nonlinear surface gravity waves is described by a kinetic equation of the kind

$$\frac{\partial n(\boldsymbol{k}_4)}{\partial t} = \int |M_{1,2,3,4}|^2 \delta(\omega_1+\omega_2-\omega_3-\omega_4)\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4) \times [n_1 n_2(n_3+n_4)-n_3 n_4(n_1+n_2)]\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 \equiv I. \quad (1)$$

Here $\boldsymbol{k}_i$ is the wavenumber vector, $\omega_i \equiv \omega(\boldsymbol{k}_i) = (gk)^{1/2}$ is the frequency of surface gravity waves with wavenumber k, $n_i = n(\boldsymbol{k}_i)$ is a wave-action spectrum, connected to a surface elevation energy spectrum $S(\boldsymbol{k})$ by the relation

$$n(\boldsymbol{k}) = 2\pi^2 g S(\boldsymbol{k})/\omega; \quad (2)$$

$M_{1,2,3,4} \equiv M(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ are the matrix elements for nonlinear interactions of surface gravity waves, given, for example, in Crawford, Saffman & Yuen (1980); g is the acceleration due to gravity, and $\delta(\ldots)$ are the delta functions.

By means of a scaling analysis, Zakharov & Filonenko (1966, hereafter referred to as ZF) found analytically that a number of stationary solutions of (1) exist in an infinite frequency range $0 < \omega < \infty$. For the case of an angular isotropic spectral density distribution in $\boldsymbol{k}$-space, these solutions are as follows (Zakharov & Zaslavskii 1982):

$$n(\boldsymbol{k}) = \text{const}, \quad n(\boldsymbol{k}) \sim \omega^{-1}, \quad n(\boldsymbol{k}) \sim \omega^{-23/3}, \quad n(\boldsymbol{k}) \sim \omega^{-8}. \quad (3a\text{–}d)$$

The spectra (3*a*, *b*) correspond to a homogeneous distribution of an action spectral density and an energy spectral density over $\boldsymbol{k}$-space, respectively. They make the interaction integral I on the right-hand side of (1) identically zero. This means that these spectra describe an equilibrium ('no-flux') distribution of spectral densities $n(\boldsymbol{k})$ and $S(\boldsymbol{k})$ over $\boldsymbol{k}$-space in any frequency interval. By analogy with thermodynamics, such spectra may be called conventionally 'thermalized' ones. In the frequency representation the spectra (3*a*, *b*) correspond to the relationships

$$n(\omega) \sim \omega^3, \quad n(\omega) \sim \omega^2, \quad (4a, b)$$

which follow from the relation $n(\boldsymbol{k}) = n(\omega, \theta) g^2/2\omega^3$ and the angular isotropy

condition. Owing to the obvious divergence of the integral I for such $n(\omega)$ functions, spectra (4) have no physical meaning (Zakharov & Zaslavskii 1982).

From direct calculations (Polnikov 1989) it is known that spectra (3c, d) do not make the integral I equal to zero identically in a finite frequency interval. This means that in any frequency interval certain fluxes of wave action F_n and energy F_ε through a spectrum exist due to nonlinear interactions. These fluxes are connected with the rates of nonlinear transfers $\partial n/\partial t$ and $\partial S/\partial t$ by the relationships

$$\partial n(\boldsymbol{k})/\partial t + \nabla_k F_n(\boldsymbol{k}) = 0, \quad \partial S(\boldsymbol{k})/\partial t + \nabla_k F_\varepsilon(\boldsymbol{k}) = 0. \tag{5a, b}$$

In the stationary case, the relevant fluxes should be constant over the wavenumber in $\boldsymbol{k}$-space (or over the frequency in the ω-space). For this reason, spectra (3c, d) are called flux (or fluxable) ones. According to Zakharov & Zaslavskii (1982), spectrum (3c) corresponds to a constant flux of a wave action F_n towards the low frequencies (flux 'down') and spectrum (3d) to the constant flux of energy F_ε towards high frequencies (flux 'up').

In a frequency representation, the flux spectra are described by the relationships

$$S(\omega) = c_1 F_n^{1/3} g \omega^{-11/3}, \quad S(\omega) = c_2 F_\varepsilon^{1/3} g^{4/3} \omega^{-4}, \tag{6a, b}$$

where c_1, c_2 are dimensionless constants of order unity. For a long time, spectra (6) have been considered as the theoretical basis for experimental fully developed wind waves spectra. However, a direct numerical confirmation of the possibility of their existence in a finite frequency interval as a stationary solution of equation (1), complemented by a source $G(\omega) > 0$ and a sink $D(\omega) < 0$ at each end of the frequency interval, has not been achieved. Our work is devoted to the solution of this problem.

With the aim of clarifying only the general features of the above problem, we shall consider in this paper the simplest case of the isotropic angular distribution of spectral density corresponding to the theory of ZF. The more general (anisotropic) case will be considered in a future work.

2. Calculation technique

The following equation is solved numerically:

$$\partial S(\omega, \theta)/\partial t = I + G + D. \tag{7}$$

Here I is the nonlinear scattering integral written in the frequency–angular energy spectrum $S(\omega, \theta)$ representation, $G(\omega) > 0$ is an energy source function, located at one end of the frequency interval used, and $D(\omega) < 0$ is an energy sink, located at the other end of the same interval.

To calculate the integral I, we have used a discrete frequency–angle grid $\{\omega_i, \theta_j\}$ given by $\omega_i = 0.5(1.12)^i$ $(i = 1, \ldots, 18)$ and $\theta_j = -\pi + \pi j/24$ $(j = 0, \ldots, 23)$ (see the frequency set in figures 1–4). A larger integration grid for I would take much more computation time.

Our method of calculation of the kinetic integral is similar to that derived by Masuda (1980). It is sufficiently stable and accurate for our aims. A detailed description of it is given in Polnikov (1989).

The solution of equation (7) is found by the Euler method with two-step smoothing of the spectrum according to

$$S_{n+1}(\omega, \theta) = \tfrac{1}{2}(S_n + \{S_n + [I(S_n) + (G + D)]\,\Delta t_n\}), \tag{8}$$

where S_n and S_{n+1} are the spectrum values for two consecutive time steps and Δt_n is the relevant time increment. The choice of Δt_n is made automatically from the

condition that the spectrum increment must not be more than 50 % of the current value $S(\omega, \theta)$ at any point of the grid $\{\omega_i, \theta_j\}$.

Another important feature of the calculation technique is the substitution of an exact spectrum value at the irregular ω, θ space points, provided by the δ-functions in the integrand of I, by appropriate interpolation of spectrum values from neighbouring regular points of the grid $\{\omega_i, \theta_j\}$. The replacement is achieved by using

$$S(\omega, \theta) = \hat{S}(\hat{\omega}, \hat{\theta})(1-\Delta\omega-\Delta\theta)+\hat{S}(\tilde{\omega}, \hat{\theta})\Delta\omega+\hat{S}(\hat{\omega}, \tilde{\theta})\Delta\theta, \tag{9}$$

derived in Polnikov (1990) which is devoted to the numerical solution of the kinetic equation (1). In (9) the following notation is used: $\hat{S}$ is the previously calculated spectra value, $\hat{\omega}$ and $\hat{\theta}$ are the coordinates of the grid points which are nearest to the given ω and θ, $\tilde{\omega}$ and $\tilde{\theta}$ are the next-neighbour points, such that ω is located between $\hat{\omega}$ and $\tilde{\omega}$ and θ between $\hat{\theta}$ and $\tilde{\theta}$. The magnitudes of $\Delta\omega$ and $\Delta\theta$ are defined by

$$\Delta\omega = (\omega-\tilde{\omega})/|\hat{\omega}-\tilde{\omega}|, \quad \Delta\theta = (\theta-\tilde{\theta})/|\hat{\theta}-\tilde{\theta}|. \tag{10a, b}$$

The use of (9) and (10) allows numerical solution of the kinetic equation without loss of accuracy, and also a large-time evolution of the spectral shape to be investigated (Polnikov 1990).

The main difficulty in the numerical solution of equation (7) (or (1)) arises from the strong dependence of relaxation time τ_r, determined by the relationship

$$\tau_r = S(\omega)/(\partial S(\omega)/\partial t), \tag{11}$$

on frequency ω. According to Resio & Perrie (1991), the relevant dependence is $\tau_r \sim \omega^{-3}$. Thus using different values of Δt for different frequencies may be the optimal way to proceed. However, in our numerical solution we use a constant value of Δt for all frequencies, and so in the numerical solution of (7) it is necessary to smooth the high-frequency spectrum tail in order to prevent fast high-frequency oscillations of the spectral intensity (Polnikov 1990). For the same reason, the spectrum shape at some time moments appears not to be very smooth.

As a result of numerous preliminary calculations, a method of choosing an adaptive Δt value was derived that permitted the above difficulty to be overcome. In addition, a special procedure for smoothing high-frequency oscillations of the spectral density was developed to allow the smooth rise of a numerical solution of (7) to a steady state.

The smoothing is carried out in the following way. For a spectrum value near the upper end of the frequency interval ($\omega > \omega_b$), we use the following set of two consecutive restrictions:

$$S(\omega) = \max\,[S(\omega), 0.2S(\omega_b)(\omega/\omega_b)^n], \tag{12a}$$

$$S(\omega) = \min\,[S(\omega), 3.0S(\omega_b)(\omega/\omega_b)^n]. \tag{12b}$$

Here $\omega_b \equiv \omega_{13} \approx 2.2$ is a conventional boundary of the spectrum tail and $n = 4$. Owing to the quite wide range of spectrum values permitted by (12*a*, *b*), the actual solution of (7) is not damaged and the accuracy of solution is good enough.

To illustrate the results obtained, four of the calculation runs were chosen, having parameters presented in table 1. $S(\omega)$, $G(\omega)$ and $D(\omega)$ are given in SI units. The real magnitudes of the frequency and spectra $S(\omega, \theta)$ are defined by presetting the frequency scale. In our case, this scale is given in units of radian per second (rad s^{-1}).

The values of $G(\omega)$ and $D(\omega)$ are related to each other in that they produce the same flux F_e:

$$F_e = \int_0^\infty G(\omega)\,\mathrm{d}\omega = \int_0^\infty D(\omega)\,\mathrm{d}\omega. \tag{13}$$

Run	Initial spectrum $S(\omega,\theta)$ ($m^2 s^{-1}$)	Location and value of G	Location and value of D
1	$45/\omega^6$	0.1 at $2.7 < \omega < 3.1$	-0.4 at $0.5 < \omega < 0.7$
2	$75/\omega^8$	0.8 at $2.7 < \omega < 3.1$	-3.2 at $0.5 < \omega < 0.7$
3	$3/\omega^4$	0.4 at $0.5 < \omega < 0.7$	-0.1 at $2.7 < \omega < 3.1$
4	$3/\omega^4$	3.2 at $0.5 < \omega < 0.7$	-0.8 at $2.7 < \omega < 3.1$

TABLE 1. Run parameters in the solution of equation (7). The upper end of the frequency band was 3.84.

The choice of values of G and D and their location are the theoretical idealization needed for a comparison of theory ZF and our calculations. Run 2 differs from run 1 in that G and D values are increased by a factor of 8. The difference between runs 3 and 4 is analogous. Such a choice is made in order to find the dependence of a flux spectrum level on the values of fluxes F_e and F_n.

The results obtained demonstrate the existence of stationary solutions of (7), their dependence on the values of G and D (or, more exactly, on values of fluxes F_n and F_e) and their independence of the initial spectral form.

3. Results and analysis

One unexpected result of the preliminary calculations is the strong dependence of the integral estimation features for the spectrum of the form $S(\omega,\theta) \sim \omega^{-11/3}$ on the choice of frequency integration limits. It means that for such a slowly falling spectrum, the value of the integral I is ill-defined. There was a danger of inconsistency between spectra of the forms $S(\omega) \sim \omega^{-11/3}$ and $S(\omega) \sim \omega^{-4}$ to be used for the numerical investigation at a finite frequency range. This circumstance has determined the core of the problem under consideration – to clarify the possibility of the existence of flux spectra at a finite frequency interval through a numerical solution of equation (7). This problem has not yet been considered in the scientific literature.

The results of calculations for the first run are shown in figure 1. From figure 1 one can see the process of stationary spectrum formation. With the aim of a convenient estimation of the law of spectrum dependence on frequency, in figure 1 the calculated values $\log[S(\omega)\omega^{11/3}]$ versus a logarithmic scale of frequency are plotted.

The estimated time of stabilization is of the order of $\tau_s \approx 4 \times 10^3$ s^{-1}. In the frequency range under analysis, $0.8 < \omega < 2.7$ rad s^{-1} located between the source and the sink, the form of the stationary spectrum $S(\omega)$ is described by the relationship

$$S(\omega) = 820\omega^{-3.8}\ \mathrm{m^2\ s^{-1}}. \tag{14}$$

The parameters in (14) were obtained by the root-mean-square method with statistical errors of the order of 5%. With this accuracy, the current spectrum shape $S(\omega, t)$ is close to the representation (14).

It can be seen that the result obtained accords well with the ZF theory and the spectrum (14) may be a flux spectrum with a constant value of wave-action flux to low frequencies. The discrepancy between (14) and (6*a*) results from the technical limitations of the numerical calculations described above (sparse grid, single Δt for all frequencies, small frequency interval).

Let us estimate some other parameters of the spectrum (14). The constant flux of

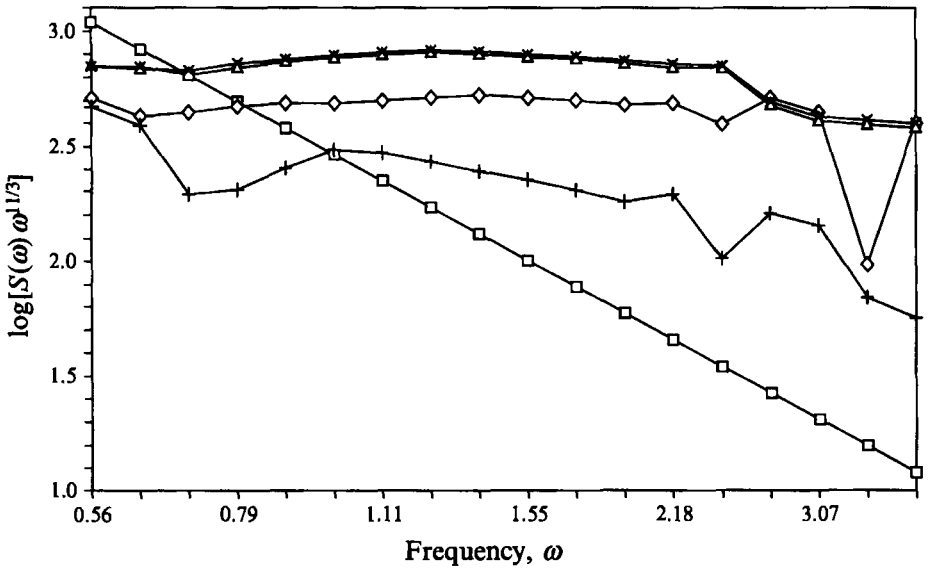

FIGURE 1. Results of the numerical solution of equation (7) for run 1 (flux down) at times: □, $t_1 = 0$; +, $t_2 = 1000$; ◇, $t_3 = 2000$; △, $t_4 = 4000$; x, $t_5 = 6000$ (in s).

wave action F_n, provided by the source G, taking into account the relation between $n(\omega)$ and $S(\omega)$, is defined by

$$F_n = 2\pi^2 \int_0^\infty \int_0^{2\pi} \frac{G(\omega, \theta)}{\omega} \, d\omega \, d\theta \approx 25 \text{ m}^3 \text{ s}^{-2}. \tag{15}$$

A comparison of (14), (15) and (5*a*) gives the estimate $c_1 \approx 30$.

The time of stabilization can be estimated from

$$\tau_s \approx \int_0^\infty \int_0^{2\pi} \frac{n(\omega, \theta) \, d\omega \, d\theta}{F_n} \approx 5 \times 10^3 \text{ s}^{-1}. \tag{16}$$

This estimate of τ_s is in a good agreement with that obtained from numerical calculations. Consequently, the parameters of the stationary solution of (7) are self-consistent.

The calculation results for run 2 are presented in figure 2. For this case, the magnitude of flux F_n is increased by a factor 8 with the aim of determining the law of the dependence of a stationary value of $S(\omega)$ on the value of F_n. As seen from figure 2, the calculated spectral form is close to the theoretical one (6*a*) and is independent of the initial spectral form. With the above accuracy the stationary spectrum is described by the relationship

$$S(\omega) = 1.65 \times 10^3 \, \omega^{-3.75} \text{ m}^2 \text{ s}^{-1}. \tag{17}$$

Here the coefficient c_1 is unchanged because the magnitude of the spectral level is increased in accordance with the increase in the values of $F_n^{1/3}$, i.e. by a factor of two.

Thus, we can state that in the course of the numerical solution of (7), the relationship $S(\omega) \sim F_n^{1/3}$ is actually realized and the spectrum, formed under the condition that the location of the source G is higher than the sink D on the frequency axis, is really a fluxable one of the 'flux-down' kind.

Consider now the 'flux-up' case (runs 3 and 4). The relevant results are shown in figures 3 and 4, where the vertical axis is given in units of $\log[S(\omega)\omega^4]$.

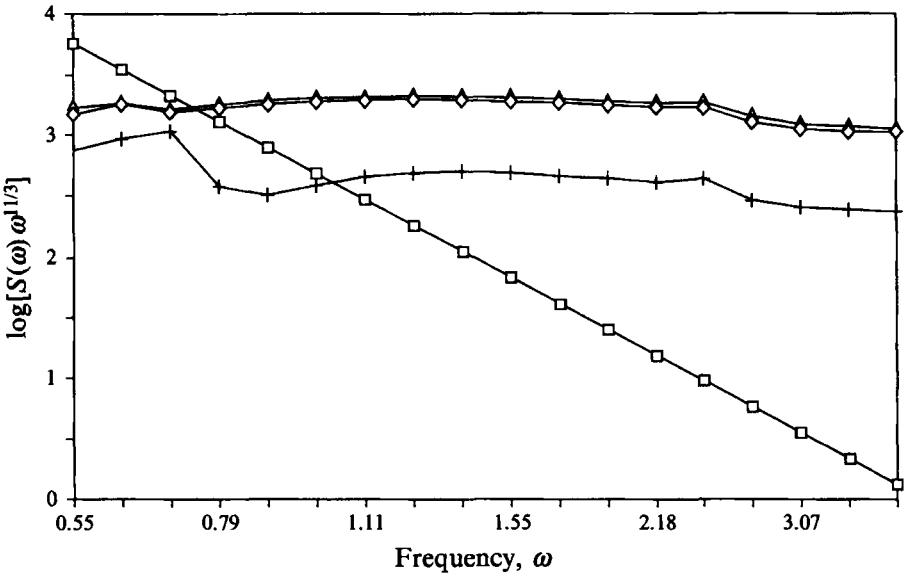

FIGURE 2. Results of the numerical solution of equation (7) for run 2 (flux down) at times: □, $t_1 = 0$; +, $t_2 = 500$; ◇, $t_3 = 1500$; △, $t_4 = 2500$.

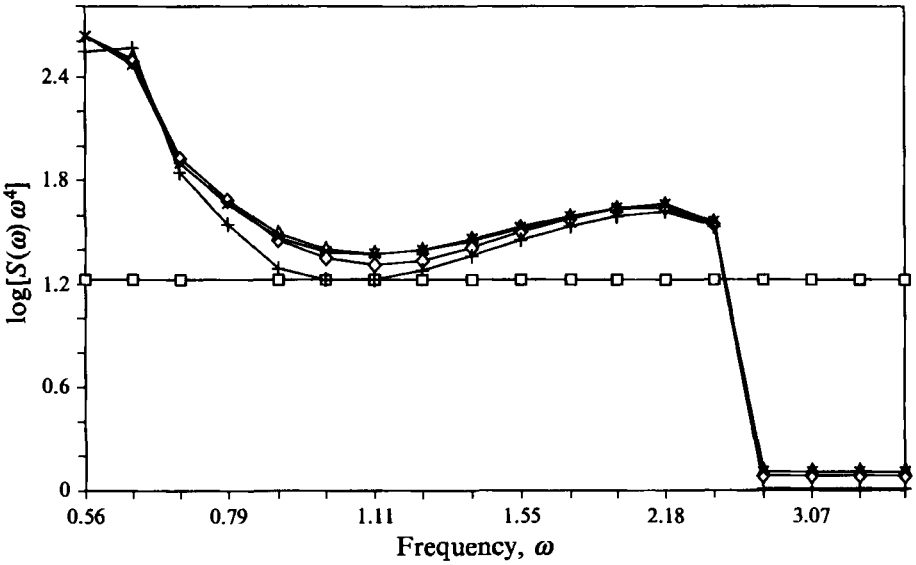

FIGURE 3. Results of the numerical solution of equation (7) for run 3 (flux-up) at times: □, $t_1 = 0$; +, $t_2 = 2000$; ◇, $t_3 = 4000$; △, $t_4 = 6000$ (in s).

The process of energy transfer to high frequencies is characterized by smooth current spectra at every moment of time. This is due to the strong dependence of the relaxation time τ_r on the frequency. For this reason, the discrete portions of energy transferred from the source to the sink by nonlinear interactions are rapidly smoothed.

For run 3, the calculated form of the stationary spectrum for the frequency band $0.8 < \omega < 2.4$ rad s^{-1} is described by the relationship

$$S(\omega) = 36 \times \omega^{-4.1} \text{ m}^2 \text{ s}^{-1}, \tag{18}$$

with an accuracy of the order of 7%. Equation (18) is very close to the theoretical one (6*b*).

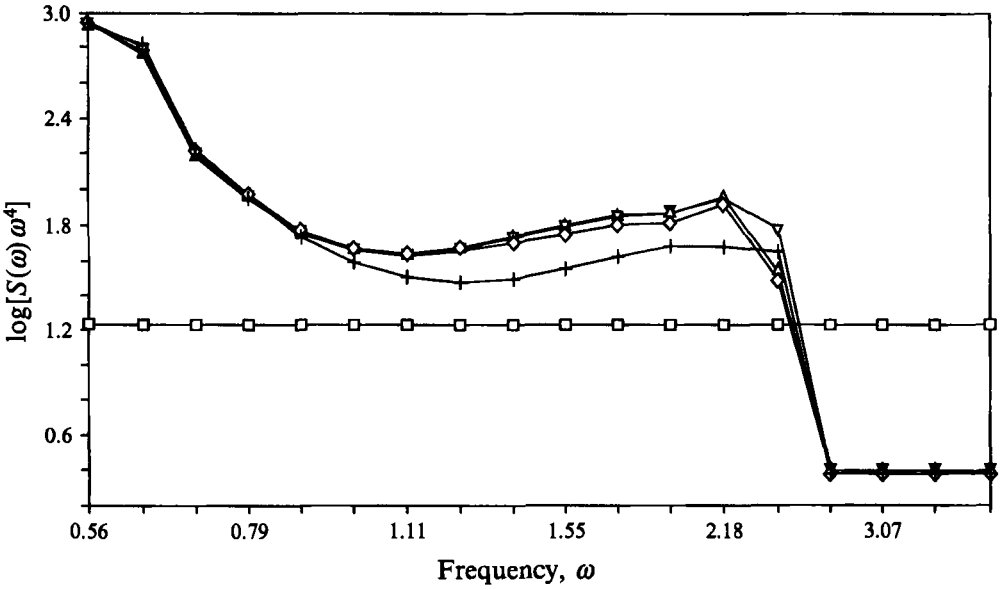

FIGURE 4. Results of the numerical solution of equation (7) for run 4 (flux-up) at times: □, $t_1 = 0$; +, $t_2 = 800$; ◇, $t_3 = 1200$; △, $t_4 = 1600$; ▽, $t_5 = 2000$.

Estimating the energy flux F_e by using a formula similar to (15), we obtain

$$F_e = \int_0^\infty \int_0^{2\pi} G(\omega, \theta)\, \mathrm{d}\omega\, \mathrm{d}\theta \approx 0.2\ \mathrm{m^2\ s^{-1}}. \tag{19}$$

From (18), (19) and (6*b*) comparison of the spectra (18) and (6*b*) gives a value $c_2 \approx 3$.

The stabilization time τ_s, defined by

$$\tau_s^t \approx \left(\int_0^\infty \int_0^{2\pi} S(\omega, \theta)\, \mathrm{d}\omega\, \mathrm{d}\theta\right) \Big/ F_e = c_2 F_e^{-2/3} g^{4/3} \omega_{\min}^{-3} \approx 3 \times 10^3\ \mathrm{s^{-1}}, \tag{20}$$

is in a good agreement with the calculated one $\tau_s^c \approx 4 \times 10^3\ \mathrm{s^{-1}}$. Some discrepancy between τ_s^t and τ_s^c may be due to the fact that only a part of the total flux F_e, determined by (19), is directed to high frequencies, with the other part of F_e going low frequencies.

With the aim of finding the law of the dependence of $S(\omega)$ on F_e for the case considered in run 4, the values of F_e was increased by a factor of 8. The results for run 4, shown in the figure 4, are in a good agreement with the theory. In fact, the height of the stationary spectrum has increased twofold, but the stabilization time was decreased by a factor of 4. Thus we can state that when the source is located lower than the sink, the 'flux-up' spectra are actually realized.

4. Conclusions

For the case of angular isotropic spectra the results presented here are in a good agreement with the known theoretical derivations by Zakharov & Filonenko (1966) and Zakharov & Zaslavskii (1982). For the first time we have numerically established that the flux spectra over a finite frequency interval are realized in the course of the solution of kinetic equation (7) as a result of time averaging of current spectra.

The necessity of averaging arises because the spectra (6) do not make integral I identically zero. For this reason the process of the formation of flux spectra occurs by

transferring a discrete amount of energy (or action) from the source to the sink. After completion of the transfer of energy, the form of the spectrum is close to the theoretical one. The error in the calculations arises from the discrepancy between the current and averaged spectra. In our calculations it is of the order of 5–7%.

The direct verification of the invariability of the flux values is difficult because of the need to integrate the nonlinear transfer at every step of the calculation and then to average it. There is a simpler way to check whether constant flux exists which will be mentioned below. Here we should pay attention to whether a stationary spectrum exists over a finite frequency interval. The stabilizing effect is possibly due to deviations of the real shapes of spectra from ideal theoretical ones.

Regarding flux evaluation the results of the recent paper by Resio & Perrie (1991) are rather interesting. Using the example of direct calculations of energy fluxes F_ε they have shown that nonlinear interactions between waves lead to both 'up' F_ε^+ and 'down' F_ε^- fluxes simultaneously. (It is obvious here that the fluxes correspond to the particular wave-action fluxes F_n^+ and F_n^-.) The total fluxes determined by (5) represent the sum of both kinds of fluxes.

On the other hand, it is not difficult to show that for any spectral shape both positive and negative values of the total flux exist (i.e. the latter are not constant). For this reason, it is very important to define what kinds of fluxes are conserved in the cases considered by means of direct calculations. In Resio & Perrie (1991), some points of the problem were clarified. A comprehensive investigation is needed, however, and we will present one in a forthcoming paper.

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